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Even-visiting random walks: exact and asymptotic results in one dimension

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Abstract

We reconsider the problem of even-visiting random walks in one dimension. This problem is mapped onto a non-Hermitian Anderson model with binary disorder. We develop very efficient numerical tools to enumerate and characterize even-visiting walks. The number of closed walks is obtained as an exact integer up to 1828 steps, i.e. some 10^{535} walks. On the analytical side, the concepts and techniques of one-dimensional disordered systems allow one to obtain explicit asymptotic estimates for the number of closed walks of $4k$ steps up to an absolute prefactor of the order of unity, which is determined numerically. All the cumulants of the maximum height reached by such walks are shown to grow as $k^{1/3}$, with exactly known prefactors. These results illustrate the tight relationship between even-visiting walks, trapping models and the Lifshitz tails of disordered electron or phonon spectra.

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1. Introduction

Random walks are one of the most ubiquitous concepts of statistical physics. The large-time behaviour of random walks is rather insensitive to details such as the presence of an underlying lattice or of an elementary time step, so that a vast range of them belongs to the universality class of Brownian motion, their continuum limit being described by the diffusion equation.

Recently, in the context of surface growth, Noh *et al* [1] have introduced a model of random walks with the constraint that every visited site should be visited an even number of times. Such walks have been named even-visiting walks [2, 3]. It has been argued that the non-local constraint of being even-visiting changes the universality class of random walks, with the typical extension of a walk of n steps scaling as $n^{1/(d+2)}$ in dimension d, instead of the usual $n^{1/2}$.

Derrida's argument [4] for this behaviour, to be recalled just below, demonstrates a deep analogy between the even-visiting walk problem, the trapping problem [5, 6] of a diffusive

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particle which is absorbed by traps located at random positions and the Lifshitz tails [7] of the density of states of electron or phonon spectra of disordered solids near band edges.

Consider for definiteness a random walk of n steps on the hypercubic lattice in dimension d, with unit lattice spacing. The probability that this walk is confined within a sphere with radius $R \ll n^{1/2}$ scales as

$$p(R) \sim \exp\left(-\frac{j^2\, n}{2d\, R^2}\right)$$

where j is the first positive zero of the Bessel function $J_{(d-2)/2}$, so that j^2/R^2 is the lowest eigenvalue of the Laplace operator in the sphere, with Dirichlet boundary conditions. Then, if $n \gg R^d$, most sites in the sphere must have been visited many times. Hence, to a good approximation, the probability that any given site has been visited an even number of times is $\frac{1}{2}$, independently of other sites. Thus, among all walks that remain within the sphere, the fraction of even-visiting ones is about

$$f(R) \sim \exp\left(-\Omega \ln 2\, R^d\right)$$

where $\Omega = \pi^{d/2}/\Gamma(d/2+1)$ is the volume of the unit ball. Consequently, for large n, the main contribution to the number of even-visiting walks comes from 'Lifshitz spheres', i.e. optimal spheres whose radius R_{L} maximizes the product $p(R)f(R)$, hence

$$R_{\mathrm{L}} \approx \left(\frac{j^2\, n}{d^2 \Omega \ln 2}\right)^{1/(d+2)}. \tag{1.1}$$

Both inequalities needed to justify the argument are satisfied by this solution, in any dimension d. Equation (1.1) gives an estimate for the typical maximum extent of an n-step even-visiting walk from its origin. The total number of such walks scales as $(2d)^n p(R_{\mathrm{L}}) f(R_{\mathrm{L}})$, hence

$$\mathcal{N}(n) \sim (2d)^n \exp\left(-\frac{d+2}{2}(\Omega \ln 2)^{2/(d+2)} \left(\frac{j^2\, n}{d^2}\right)^{d/(d+2)}\right). \tag{1.2}$$

In this paper we reconsider the one-dimensional case [1–3] in detail. We map the even-visiting walk problem onto a model with quenched disorder, analogous to a non-Hermitian Anderson model. Many concepts and techniques of one-dimensional disordered systems are then available [8–10]. We develop efficient numerical tools to enumerate even-visiting walks. The parameter in the generating function of the walks governs the strength of disorder. The disordered system comes with an invariant measure, whose support is bounded for small disorder, and becomes unbounded at some critical point, at which the generating function of even-visiting walks develops a singularity. We first compute the escape probability (4.9), which is similar to the integrated density of states of the Anderson model. This allows one to take advantage of the existence of a simpler problem, the random-mirror problem, for which many quantities can be computed analytically, and then translated in terms of even-visiting walks. Numerical checks show that the analogy between both problems indeed works at a quantitative level. We thus obtain, in particular, the estimate (6.11) for the number of closed even-visiting walks, and equations (7.3) and (7.4) for the cumulants of the maximum height reached by such a walk.

2. Definitions and mapping to a disordered system

An n-step walk on a graph is a sequence $v_0 v_1 \cdots v_n$ of vertices of the graph such that $v_0 v_1, \ldots, v_{n-1} v_n$ are edges of the graph. An even-visiting walk is such that for each vertex

$v \neq v_0, v_n$ of the graph, the number of indices i such that $v = v_i$ is even. Note that this is a non-local constraint on the walk, so that it is not unlikely that statistical properties of even-visiting random walks are rather different from those of standard random walks.

In this paper we shall concentrate on one-dimensional graphs, namely the segments with vertices $0, 1, \ldots, N$ or the half-line with vertices $0, 1, 2, \ldots$. In each case, the edges connect nearby integers.

We shall count even-visiting closed walks starting and ending at the origin (vertex 0). This problem can be reformulated as a one-dimensional disordered system [9, 10]. Consider walks on a half-line with vertices numbered $0, 1, 2, \ldots$. Each walk comes with a weight, which is computed as follows: each traversed edge contributes a factor $\sqrt{t}$, and each arrival at site i contributes a factor w_i for $i = 0, 1, 2, \ldots$. We consider for the time being that t and $w_0, w_1, \ldots$ are commuting indeterminates.

Let $F(t, w_0, w_1, \ldots)$ be the generating function for walks P starting and ending at the origin, each walk being counted with its weight. Formally,

$$F(t, w_0, w_1, \ldots) = \sum_P \sqrt{t}^{S(P)} w_0^{V_0(P)} w_1^{V_1(P)} \ldots$$

where $S(P)$ is the number of steps, and $V_i(P)$ the number of arrivals to site i. The only walk with zero steps has weight 1. Any other walk can be decomposed uniquely as a succession of elementary blocks of the following type: a step from site 0 to site 1, giving weight $\sqrt{t}\, w_1$, a closed walk on the half-line made of the sites $1, 2, \ldots$, contributing to $F(t, w_1, w_2, \ldots)$, and a step from site 1 to site 0, giving weight $\sqrt{t}\, w_0$. So

$$F(t, w_0, w_1, \ldots) = 1 + \sqrt{t}\, w_1 F(t, w_1, w_2, \ldots)\sqrt{t}\, w_0 + \left(\sqrt{t}\, w_1 F(t, w_1, w_2, \ldots)\sqrt{t}\, w_0\right)^2 + \cdots.$$

A resummation of the geometric series gives

$$F(t, w_0, w_1, \ldots) = \frac{1}{1 - t w_0 w_1 F(t, w_1, w_2, \ldots)} \tag{2.1}$$

and leads to a continued-fraction expansion

$$F(t, w_0, w_1, \ldots) = \cfrac{1}{1 - \cfrac{t w_0 w_1}{1 - \cfrac{t w_1 w_2}{1 - \cdots}}}. \tag{2.2}$$

These formulae are the starting point of all further considerations. Note that, in the expansion of $F(t, w_0, w_1, \ldots)$, the even-visiting random walks correspond exactly to monomials such that $V_1(P), V_2(P), \ldots$ are all even. Then $V_0(P)$ as well, because $V_0(P) + V_1(P) + \cdots$ is the number of steps, which is even for any closed walk. So, if the weights w_i are turned into independent random variables with vanishing odd moments and all even moments equal to 1, the average of $F(t, w_0, w_1, \ldots)$, denoted by $\overline{F(t, w_0, w_1, \ldots)}$, is the generating function for even-visiting walks on the half-line. In the same way, for finite segments, $w_0, \ldots, w_N$ are chosen as above, and $w_{N+1} = 0$.

Now, a random variable with vanishing odd moments and all even moments equal to 1 is simply a random sign: it takes values ± 1 with probability $\frac{1}{2}$. The above formulation can therefore be further simplified: if $w_0, w_1, \ldots, w_N$ are independent random signs, then so are $\varepsilon_N = w_0 w_1, \varepsilon_{N-1} = w_1 w_2, \ldots, \varepsilon_1 = w_{N-1} w_N$.

The generating series of even-visiting walks on the segment $0, \ldots, N$ reads therefore

$$F_N(t, \varepsilon_1, \ldots, \varepsilon_N) = \cfrac{1}{1 - \cfrac{t\varepsilon_N}{1 - \cfrac{t\varepsilon_{N-1}}{1 - \cfrac{\cdots}{1 - t\varepsilon_1}}}}. \tag{2.3}$$

It is clear from this expression that averages over the quenched disorder represented by the ε_is will be even functions of t. As each power of t counts for two steps, the length (number of steps) of closed even-visiting walks is a multiple of four.

Equation (2.3) implies the recursion formula

$$F_N(t, \varepsilon_1, \ldots, \varepsilon_N) = \frac{1}{1 - t\varepsilon_N F_{N-1}(t, \varepsilon_1, \ldots, \varepsilon_{N-1})}. \tag{2.4}$$

If we write $F_N = \Psi_N/\Psi_{N+1}$, with $\Psi_0 = \Psi_1 = 1$, then we have

$$\Psi_{N+1} - \Psi_N + t\varepsilon_N \Psi_{N-1} = 0. \tag{2.5}$$

This three-term linear recursion relation defines a non-Hermitian one-dimensional Anderson model with off-diagonal binary disorder. In this context, t can be interpreted as the strength of the disorder. The variables F_N, which obey the recursion relation (2.4), are the associated Riccati variables [10, 11].

It is clear that the t-expansions of $F_N(t, \varepsilon_1, \ldots, \varepsilon_N)$ and $F_{N+1}(t, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N)$ coincide up to order N (a closed walk of $2N$ steps cannot visit vertices higher than N), so all formulae have a well defined $N \to \infty$ limit.

The t-expansion of F_N starts with 1, so its power F_N^α is well defined as a formal power series in t, for α an arbitrary complex number, or even indeterminate. Surely, only when $\alpha = 1$ does this quantity have a simple interpretation as a counting problem. Nevertheless, considering arbitrary α also has its own interest (see section 8). Anyway, we set

$$f_N(t, \alpha) = \overline{F_N^\alpha(t)} \qquad f(t, \alpha) = \lim_{N\to\infty} f_N(t, \alpha).$$

3. Weak-disorder expansion

If we formally expand $F_N^\alpha(t)$ from equation (2.4) in powers of ε_N, we find

$$F_N^\alpha(t) = 1 + \sum_{m\geqslant 1} \varepsilon_N^m F_{N-1}^m t^m \frac{\Gamma(\alpha + m)}{\Gamma(\alpha) m!}$$

so that after averaging (and with $m = 2n$)

$$f_N(t, \alpha) = 1 + \sum_{n\geqslant 1} f_{N-1}(t, 2n) t^{2n} \frac{\Gamma(\alpha + 2n)}{\Gamma(\alpha)(2n)!}$$

and for $N \to \infty$

$$f(t, \alpha) = 1 + \sum_{n\geqslant 1} f(t, 2n) t^{2n} \frac{\Gamma(\alpha + 2n)}{\Gamma(\alpha)(2n)!}. \tag{3.1}$$

Let $\mathcal{N}_k$ (respectively, $\mathcal{N}_{N,k}$) denote the number of even-visiting closed walks of $4k$ steps starting at the origin on the half-line (respectively, on the segment $[0, N]$), and let us introduce the t-expansions

$$f_N(t, \alpha) = \sum_{n\geqslant 0} f_{N,2n}(\alpha) t^{2n} \qquad f(t, \alpha) = \sum_{n\geqslant 0} f_{2n}(\alpha) t^{2n}. \tag{3.2}$$

Equation (3.1) yields the following recursion relation:

$$f_{N,2m}(\alpha) = \sum_{n=1}^{m} f_{N-1,2m-2n}(2n)\frac{\Gamma(\alpha+2n)}{\Gamma(\alpha)(2n)!}$$

with initial conditions $f_{N,0}(\alpha) = 1$ for $N \geqslant 0$. The above formula is recursive both in N and m, and it holds for $N \geqslant 1$, $m \geqslant 1$. All of the above quantities are non-decreasing functions of N. Letting $N \to \infty$, we obtain

$$f_{2m}(\alpha) = \sum_{n=1}^{m} f_{2m-2n}(2n)\frac{\Gamma(\alpha+2n)}{\Gamma(\alpha)(2n)!} \tag{3.3}$$

for $m \geqslant 1$, with the initial condition $f_0(\alpha) = 1$.

This formalism can be used to compute the number of even-visiting walks in a very efficient way. To do so, it is better to trade the variable α for integers, to compute recursively an array $f_{2m}(2\ell)$ for $0 \leqslant \ell \leqslant m \leqslant k-1$, using

$$f_{2m}(2\ell) = \sum_{n=1}^{m} f_{2m-2n}(2n)\binom{2n+2\ell-1}{2n}$$

and finally to obtain the number of closed even-visiting walks with $n = 4k$ steps on the half-line, starting at 0, as

$$\mathcal{N}_k = f_{2k}(1) = \sum_{\ell=1}^{k} f_{2k-2\ell}(2\ell). \tag{3.4}$$

The same idea can be implemented in the case of walks on a finite segment.

With a few hours of computation on a workstation, using the MACSYMA software, $\mathcal{N}_k$ has been evaluated for k up to 457, and $\mathcal{N}_{N,k}$ for $N \leqslant k \leqslant 125$. Note that $\mathcal{N}_{N,k} = \mathcal{N}_k$ for $N \geqslant k$. It is therefore possible to perform accurate investigations of the number of walks and of related quantities, such as the distribution of the maximum height (see sections 6–8).

It is obvious that $\mathcal{N}_k \leqslant 16^k$, the total number of walks of $4k$ steps. It is easy to generalize this (poor) bound to see that $f_{2k}(\alpha) \leqslant 2^\alpha 16^k$. So, the radius of convergence of $f(t,\alpha)$ in t^2 is at least $\frac{1}{16}$. In fact, for real positive α, it is exactly $\frac{1}{16}$, as can be seen from the above equations as follows. Suppose $\ell \geqslant 1$. From

$$f(t,2\ell) = 1 + \sum_{m\geqslant 1} f(t,2m)t^{2m}\binom{2\ell+2m-1}{2m}$$

we see that the t^2-expansion of

$$\left(1 - t^{2\ell}\binom{4\ell-1}{2\ell}\right) f(t,2\ell) - 1 = \sum_{m\neq 0,\ell} t^{2m}\binom{2\ell+2m-1}{2m} f(t,2m)$$

has only non-negative coefficients. Hence the same is true of the t^2-expansion of

$$f(t,2\ell) - \left(1 - t^{2\ell}\binom{4\ell-1}{2\ell}\right)^{-1}.$$

Inserting this bound for $\ell = m$ on the right-hand side of equation (3.1) shows that, for $\alpha > 0$, the t^2-expansion of

$$f(t,\alpha) - 1 - \sum_{m\neq 0} t^{2m}\left(1 - t^{2m}\binom{4m-1}{2m}\right)^{-1} \Gamma(\alpha+2m)/(\Gamma(\alpha)(2m)!)$$

has only non-negative coefficients. Note that the radius of convergence (in powers of t^2) of

$$\left(1 - t^{2m}\binom{4m-1}{2m}\right)^{-1}$$

is very close to $\frac{1}{16}$ for large m. This implies that $f(t,\alpha)$ is singular at $t^2 = \frac{1}{16}$, giving the announced value for the radius of convergence. We have not been able to refine the above bounds in a useful way, starting from the weak-disorder expansion. In the next sections we obtain much better estimates by a different method.

4. Invariant measure and escape probability

In order to get a feeling for the kind of singularities that appear at $t^2 = \frac{1}{16}$, we first evaluate a quantity apparently unrelated to the counting of even-visiting random walks, namely the probability for the random variable $F(t; \varepsilon_1, \varepsilon_2, \ldots)$ to be larger than 2:

$$P_{\mathrm{esc}}(t) = \mathrm{Prob}(F > 2).$$

This quantity, referred to as the escape probability, vanishes identically for $t^2 \leqslant \frac{1}{16}$, while it is non-zero for $t^2 > \frac{1}{16}$. It is quite analogous to the integrated density of states in the Anderson model and similar disordered spectra [8–10, 12].

For the time being, consider a fixed real positive t, and define the (complementary) distribution function

$$R_N(x) = \mathrm{Prob}(F_N > x).$$

The recursion relation (2.4) implies the following relation between the distribution functions of F_N and F_{N-1}:

$$2R_N(x) = R_{N-1}\left(\frac{x-1}{tx}\right) - R_{N-1}\left(\frac{1-x}{tx}\right) + R_{N-1}\left(-\frac{1}{t}\right) - R_{N-1}\left(\frac{1}{t}\right) + 2\Theta(-x)$$

where Θ is Heaviside's the step function:

$$\Theta(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x \geqslant 0. \end{cases}$$

For large N, $R_N(x)$ approaches a limiting distribution function $R(x)$, which defines an invariant measure $\mathrm{d}R(x)$, and obeys the Dyson–Schmidt equation [10, 11]

$$2R(x) = R\left(\frac{x-1}{tx}\right) - R\left(\frac{1-x}{tx}\right) + R\left(-\frac{1}{t}\right) - R\left(\frac{1}{t}\right) + 2\Theta(-x). \quad (4.1)$$

The fractional linear mappings involved in this equation are the reciprocals of those involved in equation (2.4).

4.1. The case $0 < t < \frac{1}{4}$

In this situation, both mappings involved in equation (2.4) (or in equation (4.1)) are hyperbolic. Let $M(t) = (1 - \sqrt{1-4t})/(2t)$ be the smallest fixed point of the mapping $x \mapsto 1/(1-tx)$, and let $m(t) = 1/(1 + tM(t))$. So, $M(t) = 1/(1 - tM(t))$ and $-1/t < 0 < m(t) < 1 < M(t) < 2 < 1/t$. The mappings $x \mapsto 1/(1-tx)$ and $x \mapsto 1/(1+tx)$ are respectively increasing and decreasing (on their intervals of continuity). Hence, if x lies in the interval $I(t) =]m(t), M(t)[$, then the same is true of $1/(1-tx)$ and $1/(1+tx)$. From $F_0(t) = 1$, an inductive argument shows that $F_N(t, \varepsilon_1, \ldots, \varepsilon_N)$ takes its values in $I(t)$ for all N. Hence the support of the invariant measure is contained in $I(t)$. In particular, we have $R(x) = 0$ for $x \geqslant M(t)$ and $R(x) = 1$ for $x \leqslant m(t)$, so that equation (4.1) can be further simplified for $x \in I(t)$ to

$$2R(x) = R\left(\frac{x-1}{tx}\right) - R\left(\frac{1-x}{tx}\right) + 1. \tag{4.2}$$

The graph of $R(x)$ is a (decreasing) devil's staircase, with plateaux at the dyadic numbers $R = m/2^n$, with $n = 1, 2, \ldots$ and $1 \leqslant m$ (odd) $\leqslant 2^n - 1$. The support of the invariant measure $\mathrm{d}R(x)$, i.e. the closure of the set of points x around which $R(x)$ is not a constant, is a Cantor set of measure zero. The heights of the plateaux do not depend on t, but their size increases monotonically with t. When $t \to 0^+$, $R(x)$ tends to the step function $\Theta(1-x)$. The other, more interesting limiting case $t = \frac{1}{4}$ is investigated in the next subsection.

4.2. The case $t = t_c = \frac{1}{4}$

In this borderline case, the mapping $x \mapsto 1/(1 - t_c x)$ is parabolic: it has a degenerate fixed point $x_c = 2$, which coincides with the upper bound of the interval $I\left(\frac{1}{4}\right) = \left[\frac{2}{3}, 2\right]$.

The behaviour of the distribution function $R(x)$ near this upper bound will be our first example of an exponentially small, Lifshitz-like singularity. For x close to 2, equation (4.2) simplifies to

$$2R(x) = R\left(\frac{4(x-1)}{x}\right). \tag{4.3}$$

The above equation is actually an identity for x greater than $\frac{6}{7}$, the image of the lower bound $\frac{2}{3}$ by the mapping $x \mapsto 1/(1 + t_c x)$. We set

$$y = \frac{2}{2-x} \qquad x = 2 - \frac{2}{y} \tag{4.4}$$

so that the mapping $x \mapsto 4(x-1)/x$ corresponds to $y \mapsto y - 1$. The general solution of equation (4.3) therefore reads

$$R(x) = 2^{-y} A(y) \tag{4.5}$$

where $A(y)$ is a bounded periodic function of its argument, with unit period. Periodic amplitudes are quite frequent in the realm of one-dimensional disordered systems [9, 10, 13]. The present situation is analogous to that of the invariant measure of the Anderson model right at a band edge [12]. Figure 1 shows a plot of the periodic amplitude $A(y)$, obtained from exactly iterating the Dyson–Schmidt equation (4.2) a large enough number of times, starting from the initial data $R_0(x) = \Theta(1-x)$.

4.3. The case $t > \frac{1}{4}$

In this situation, the mapping $x \mapsto 1/(1 - tx)$ is elliptic: its two fixed points acquire an imaginary part. With the parametrization

$$t = \frac{1}{4\cos^2\phi} \qquad (0 < \phi < \pi/2) \tag{4.6}$$

the fixed points read $x_\pm = 1 + \exp(\pm 2\mathrm{i}\phi)$. Consequently, the support of the invariant measure is unbounded. A fraction of the measure escapes above 2, hence the name 'escape probability' for the quantity $P_{\text{esc}} = R(2)$.

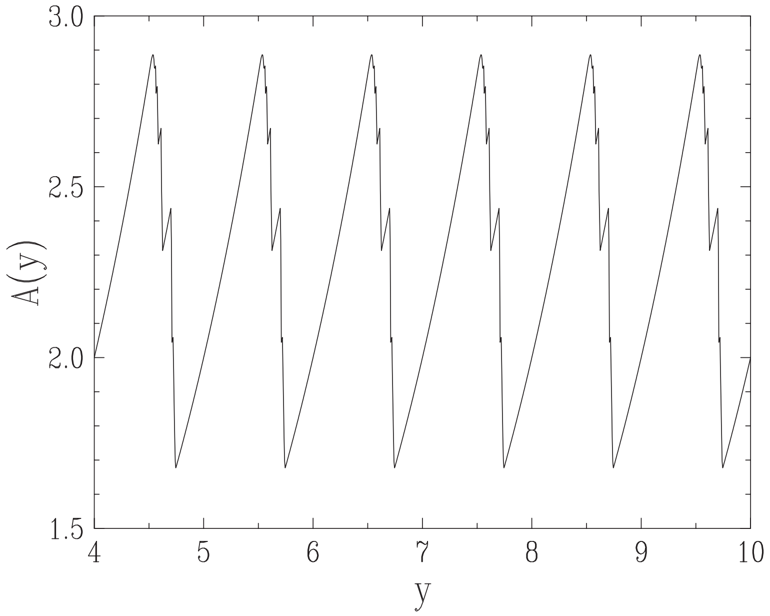

Figure 1. Plot of the periodic amplitude $A(y)$ entering expression (4.5) of the invariant measure near its upper edge in the critical case ($t = t_c = \frac{1}{4}$).

When $t - \frac{1}{4} \approx \phi^2/4$ is small, the fixed points $x_\pm \approx 2 \pm 2i\phi$ are close to $x_c = 2$, so that the escape phenomenon takes place through a narrow channel. It is therefore legitimate to deform the result (4.5) as follows. Instead of equation (4.4), we set

$$e^{2i\psi} = \frac{x - x_+}{x - x_-} \qquad x = 2\cos\phi\,\frac{\sin(\psi - \phi)}{\sin\psi} \qquad (0 < \psi < \pi) \tag{4.7}$$

so that the elliptic mapping $x \mapsto (x-1)/(tx)$ corresponds to $\psi \mapsto \psi - \phi$. The angle ratio ψ/ϕ is therefore the appropriate deformation of the coordinate y (up to an additive constant). There is actually no such constant, as equations (4.4) and (4.7) yield $\psi = y\phi + \mathcal{O}(\phi^3)$. Equation (4.5) therefore implies

$$R(x) \approx 2^{-\psi/\phi} A(\psi/\phi) \tag{4.8}$$

in the regime where ϕ is small, while the angular variable ψ is arbitrary. In particular, $x = x_c = 2$ corresponds to $\psi = \pi/2 + \phi$, hence the following prediction for the escape probability:

$$P_{\text{esc}} \approx 2^{-\pi/(2\phi)-1} A(\pi/(2\phi)) \approx \frac{1}{2}\exp\left(-\frac{\pi \ln 2}{4\sqrt{t - 1/4}}\right) A\left(\frac{\pi}{4\sqrt{t-1/4}}\right) \tag{4.9}$$

which is the result we were after. The escape probability is therefore exponentially small in $\left(t - \frac{1}{4}\right)^{-1/2}$, with an asymptotically periodic amplitude A, already entering equation (4.5), and shown in figure 1. The result (4.9) is fully analogous to the Lifshitz tail of the integrated density of states in the Anderson model and similar disordered spectra [9, 10, 12].

We have not been able to obtain by direct means estimates of the (more interesting) singularities that govern the asymptotics of the number of even-visiting random walks. However, the result (4.9) for the escape probability is quite suggestive, especially because of its formal analogy with the Lifshitz tail for the integrated density of states of electron or phonon spectra. The crucial ingredient in this result is that the transformation $x \mapsto 1/(1-tx)$ turns from hyperbolic to elliptic at $t = t_c = \frac{1}{4}$. Equivalently, the singularity (4.9) is due to the occurrence in the recursive formula (2.4) of regions made of a large number, of the order of $\pi/(2\phi)$, of consecutive positive εs, which are one-dimensional analogues of Lifshitz spheres. The other transformation $x \mapsto 1/(1+tx)$ has no impact on the exponentially small form of the singularity. Its influence is only felt on the periodic modulation.

In the next section, we introduce a simpler model, the random-mirror model, which keeps the essential feature of the original even-visiting walk problem, i.e. the transformation $x \mapsto 1/(1-tx)$, but for which quantities of interest are directly computable by elementary means. The random-mirror model is similar to the binary random harmonic chain where a finite fraction of the atoms has an infinite mass, first considered by Domb *et al* [14]. In this limiting case, the system splits into an infinite collection of independent finite molecules, so that the integrated density of states, among many other quantities, can be evaluated by simple algebra. The same simplification occurs for diffusion in the presence of perfectly absorbing sites [10]. By analogy with these situations, we claim that the random-mirror model and the even-visiting walk problem have similar exponentially small singularities. The simple and explicitly computable periodic amplitudes of the first model are just replaced by (admittedly complicated) unknown periodic functions. In the leading terms of large-order t-expansions, this replacement is just responsible for an absolute prefactor.

5. A case study: the random-mirror model

If the distribution of the variables ε_i is modified, so that they take a value 0 or 1 with respective probabilities p or $1-p$ (before they took value -1 or 1 with probability $\frac{1}{2}$), the equations obtained previously for even-visiting walks are modified in a straightforward way. We assume $p \neq 0, 1$. The physical interpretation of the present model is that any site $i = 1, 2, \ldots$ is totally reflecting with probability p, so that the particle moves freely between the origin and the first reflecting site, hence the name 'random-mirror model'.

From a more mathematical viewpoint, this amounts to replacing the map $x \mapsto 1/(1+tx)$ by the constant map $x \mapsto 1$, without changing the second map $x \mapsto 1/(1-tx)$. We shall also consider the case of a general constant map $x \mapsto X$ (instead of $x \mapsto 1$), where X is an indeterminate (most of the time specialized to a real number $X < 2$).

We shall compare in detail the random-mirror model and the even-visiting walk model, so that it is useful to change notation: for the random-mirror model we use G, g and S, instead of F, f and R.

5.1. Weak-disorder expansion

Let us first fix $X = 1$. From the recursion

$$G_N(t, \varepsilon_1, \ldots, \varepsilon_N) = \frac{1}{1 - t\varepsilon_N G_{N-1}(t, \varepsilon_1, \ldots, \varepsilon_{N_1})}$$

we obtain

$$g_N(t, \alpha) = \overline{G_N^\alpha(t)} = 1 + p \sum_{n \geqslant 1} g_{N-1}(t, n) t^n \frac{\Gamma(\alpha+n)}{\Gamma(\alpha) n!}$$

which in turn leads to a recursion relation for the coefficients of the expansion $g_N(t,\alpha) = \sum_n g_{N,n}(\alpha)t^n$, namely

$$g_{N,m}(\alpha) = p\sum_{n=1}^{m} g_{N-1,m-n}(n)\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!}$$

with initial conditions $g_{N,0}(\alpha) = 1$ for $N \geqslant 0$. For $N \to \infty$, we obtain

$$g(t,\alpha) = 1 + p\sum_{n\geqslant 1} g(t,n)t^n\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!} \tag{5.1}$$

and

$$g_m(\alpha) = p\sum_{n=1}^{m} g_{m-n}(n)\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!} \tag{5.2}$$

for $m \geqslant 1$, with the initial condition $g_0(\alpha) = 1$. We can repeat the argument given in the previous section for $f(t,\alpha)$ to show that for positive α, the t-expansion of $g(t,\alpha)$ has a radius of convergence $\frac{1}{4}$. The above formalism is easily extended to generic values of *X*. For example, equation (5.1) becomes

$$g(t,\alpha) = 1 - p + pX^\alpha + p\sum_{n\geqslant 1} g(t,n)t^n\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!}$$

while equation (5.2) remains unchanged, with the initial condition being modified to $g_0(\alpha) = 1 - p + pX^\alpha$.

5.2. *Escape probability*

We now evaluate the escape probability by more direct means than for the case of even-visiting walks. The random variable G is equal to Q_n with probability $(1-p)p^n$, where the Q_n are defined recursively as follows:

$$Q_{n+1} = \frac{1}{1 - tQ_n} \qquad Q_0 = X. \tag{5.3}$$

This definition gives a meaning to Q_n for any integer n, positive or negative, and shows that Q_n is a linear fractional function of *X*. The distribution function $S(x) = \mathrm{Prob}(G > x)$ reads explicitly

$$S(x) = (1-p)\sum_{n\geqslant 0} p^n\,\Theta(Q_n - x). \tag{5.4}$$

For the random-mirror model *stricto sensu*, we have $X = 1$. Suppose $t > \frac{1}{4}$. With the parametrization (4.6), we infer from the recursion (5.3) the explicit expression

$$Q_n = 2\left(1 - \sin\phi\,\frac{\cos(n+1)\phi}{\sin(n+2)\phi}\right). \tag{5.5}$$

Then $Q_n \geqslant 2$ is equivalent to either $\sin(n+2)\phi > 0$ and $\cos(n+1)\phi \leqslant 0$, or $\sin(n+2)\phi < 0$ and $\cos(n+1)\phi \geqslant 0$. The smallest n such that $Q_n \geqslant 2$ belongs to the first case, and reads[2] $n_0 = \lceil \pi/(2\phi)\rceil - 1$. When t is close to $\frac{1}{4}$, i.e. ϕ is small, many ns following n_0, up to $n_1 \approx 2n_0$,

[2] The symbol $\lceil x\rceil$ denotes the smallest integer larger than or equal to x, while $\lfloor x\rfloor$ denotes the largest integer smaller than or equal to x. For x not an integer, we have $\lceil x\rceil = \lfloor x\rfloor + 1$, and the fractional part of x is defined as $\mathrm{Frac}(x) = x - \lfloor x\rfloor = x - \lceil x\rceil + 1$.

have $Q_n \geqslant 2$. Consequently, $P_{\rm esc} = S(2)$ can be estimated as $(1-p)\sum_{n_0 \leqslant n \leqslant n_1} p^n$, leading to the result

$$P_{\rm esc} \approx p^{\lceil \pi/(2\phi) \rceil - 1} \tag{5.6}$$

up to a correction of the order of $\mathcal{O}(p^{\pi/\phi})$.

When the initial point is not 1, but a generic X, equation (5.5) generalizes to

$$Q_n = 2\left(1 - \sin\phi\, \frac{\cos(n+1)\phi + (1-X)\cos(n-1)\phi}{\sin(n+2)\phi + (1-X)\sin n\phi}\right). \tag{5.7}$$

The smallest n such that $Q_n \geqslant 2$ reads $n_0(X) = \lceil \pi/(2\phi) - \Delta(\phi, X) \rceil$, with

$$\Delta(\phi, X) = \frac{1}{\phi} \arctan\left(\frac{X}{2-X} \tan\phi\right). \tag{5.8}$$

Hence

$$P_{\rm esc}(X) \approx p^{\lceil \pi/(2\phi) - \Delta(\phi, X) \rceil}$$

again up to a correction of the order of $\mathcal{O}(p^{\pi/\phi})$. For small ϕ, this result can be recast in a form similar to equation (4.9), namely

$$P_{\rm esc}(X) \approx p^{\pi/(2\phi)}\, A_X(\pi/(2\phi)) \tag{5.9}$$

where the periodic amplitude $A_X(x)$ has the explicit expression

$$A_X(x) = p^{-\,{\rm Frac}(x - \Delta_0(X)) + 1 - \Delta_0(X)}$$

with

$$\Delta_0(X) = \Delta(0, X) = \frac{X}{2-X}.$$

This result shows that the random-mirror problem at $p = \frac{1}{2}$ and even-visiting walks have the same leading singularities (compare equations (5.9) and (4.9)): the escape probability of the random-mirror problem ($X = 1$), of the generalized one (X generic), and of even-visiting walks only differ by their periodic modulation. The same property is known in the case of spectra of disordered systems: details of the distribution of random masses, random site potentials, and so on, only enter the shape of the periodic amplitudes of Lifshitz tails [6, 10, 12].

6. Number of even-visiting walks

The purpose of this section is to establish the asymptotic estimate (6.11) for the total number of even-visiting walks starting and ending at the origin, using equations (3.2) and (3.4).

The escape probability $P_{\rm esc}$ was easy to estimate in the even-visiting walk problem, because it is a purely singular quantity, which vanishes as the parameter t approaches $t_c = \frac{1}{4}$. In contrast, the singularities of the functions $f(t, \alpha)$ or $g(t, \alpha)$ at $t = \frac{1}{4}$ are less easily grasped, because the latter quantities also have a regular part.

We shall first consider the random-mirror model with $X = 1$. In this case, we shall see that there is a simple relation between the discontinuity of $g(t, 1)$ for $t > \frac{1}{4}$ and the escape probability. A similar correspondence is expected to hold for the discontinuity of $f(t, 1)$, up to a periodic modulation. These discontinuities provide a direct way to estimate the large-order behaviour of t-expansions. The latter will lead us to estimate the number of even-visiting random walks (equation (6.11)).

6.1. Discontinuity of $g(t, 1)$

Let us consider first the random-mirror model in the simpler situation where $X = 1$. By analogy with equation (5.4), we have the closed formula

$$g(t,\alpha) = (1-p)\sum_{n\geqslant 0} p^n Q_n^\alpha. \tag{6.1}$$

Their recursive definition (5.3) shows that the $Q_n(t)$ are rational functions of t. Their explicit expression (5.5) yields the decomposition

$$Q_n(t) = 1 + \frac{4}{n+2}\sum_{m=1}^{\lfloor (n+1)/2\rfloor}\left(\frac{1}{1-4t\cos^2[\pi m/(n+2)]} - 1\right)\sin^2\frac{\pi m}{n+2}. \tag{6.2}$$

It is therefore clear that all the poles of the rational function $Q_n(t)$ lie on the half-line $]\frac{1}{4}, +\infty[$. We normalize the discontinuity of a real-analytic function $f(t)$ on the real axis as $D_f(t) = (\mp 1/\pi)\,\mathrm{Im}\, f(t \pm \mathrm{i}0)$, in such a way that the discontinuity of $1/t$ is exactly $\delta(t)$. Let t be real and such that $t \geqslant \frac{1}{4}$, and consider the integrated discontinuity

$$D(n,t) = \int_{1/4}^{t} D_{Q_n}(u)\,\mathrm{d}u.$$

Equation (6.2) yields

$$D(n,t) = -\frac{1}{n+2}\sum_{m=1}^{\lfloor (n+1)/2\rfloor}\Theta\left(t - \frac{1}{4\cos^2[\pi m/(n+2)]}\right)\tan^2\frac{\pi m}{n+2}. \tag{6.3}$$

Equation (6.1) implies that the integrated discontinuity of $g(t, 1)$ reads

$$D_1(t) = (1-p)\sum_{n\geqslant 0} p^n D(n,t).$$

Equation (6.3) leads to the following explicit result, in terms of ϕ:

$$D_1(\phi) = -\frac{1-p}{p^2}\sum_{n\geqslant 2}\frac{p^n}{n}\sum_{m=1}^{\lfloor (n-1)/2\rfloor}\Theta\left(\phi - \frac{\pi m}{n}\right)\tan^2\frac{\pi m}{n}$$

which we reorganize as

$$D_1(\phi) = -\frac{1-p}{p^2}\sum_{m\geqslant 1}\sum_{n\geqslant 2m-1}\frac{p^n}{n}\Theta\left(\phi - \frac{\pi m}{n}\right)\tan^2\frac{\pi m}{n}.$$

Finally, because $0 < \phi < \pi/2$, the condition $\pi m/n \leqslant \phi$ is always more stringent than $n \geqslant 2m - 1$, hence

$$D_1(\phi) = -\frac{1-p}{p^2}\sum_{m\geqslant 1}\sum_{n\geqslant \lceil \pi m/\phi\rceil}\frac{p^n}{n}\tan^2\frac{\pi m}{n}.$$

For small ϕ, the contribution of $m = 1$ is exponentially larger than the other ones, so that the asymptotic behaviour of $D_1(\phi)$ is

$$D_1(\phi) \approx -\frac{\phi^3}{\pi}\, p^{\lceil \pi/\phi\rceil - 2}. \tag{6.4}$$

The above formula comes entirely from the pole of $Q_n(t)$ closest to the critical point $t_c = \frac{1}{4}$. This gives the clue for treating the case of a general constant map. In that case, Q_n, as given by equation (5.7), has poles for $\tan(n+1)\phi/\tan\phi = -X/(2-X)$. Let $\phi_1(X, n)$ be the smallest solution of the latter equation. One checks that, for large n,

$$\phi_1(X, n) = \frac{\pi}{n} - \frac{2\pi}{(2-X)n^2} + \mathcal{O}(n^{-3}).$$

The contributions of the closest poles for each n to the total integrated discontinuity are

$$-(1-p)\sum_n \frac{\sin^2\phi_1}{\cos^3\phi_1} p^n \frac{\cos(n+1)\phi_1 + (1-X)\cos(n-1)\phi_1}{(n+2)\cos(n+2)\phi_1 + (1-X)n\cos n\phi_1}\,\Theta(\phi-\phi_1).$$

This expression looks formidable, but it simplifies drastically at small ϕ. From the definition of $\phi_1(X, n)$, the smallest n giving a non-vanishing contribution to the above sum is such that

$$\frac{\tan(n+1)\phi}{\tan\phi} \leqslant -\frac{X}{2-X} < \frac{\tan n\phi}{\tan\phi}$$

so that $n = \lceil \pi/\phi - \Delta(\phi, X)\rceil - 1$, with the definition (5.8). Hence

$$D_X(\phi) \approx -\frac{\phi^3}{\pi}\, p^{\lceil \pi/\phi - \Delta(\phi,X)\rceil - 1}$$

i.e.

$$D_X(\phi) \approx -\frac{\phi^3}{\pi}\, p^{\pi/\phi}\, B_X(\pi/\phi) \tag{6.5}$$

with

$$B_X(x) = p^{-\operatorname{Frac}(x-\Delta_0(X))-\Delta_0(X)} = \frac{A_X(x)}{p}. \tag{6.6}$$

The exponentially small factor $p^{\pi/\phi}$ in the result (6.5) for the discontinuity of $g(t, 1)$ is just the square of the corresponding factor in the result (5.9) for the escape probability. This correspondence, which holds for any value of X, is expected to also hold for the even-visiting walk problem. The simple relationship between the periodic amplitudes A_X and B_X is, however, a peculiarity of the random-mirror problem, that we do not expect to hold in general. We quote for further reference the constant Fourier component of the periodic function $B_X(x)$,

$$B_X^{(0)} = \int_0^1 B_X(x)\,\mathrm{d}x = \frac{1-p}{|\ln p|}\, p^{-\Delta_0(X)-1} = \frac{1-p}{|\ln p|}\, p^{-2/(2-X)}. \tag{6.7}$$

6.2. Asymptotic results

We again consider first the generalized random-mirror model, with arbitrary X. The knowledge of the integrated discontinuity $D_X(t)$ allows one to compute the asymptotics of the t-expansion of $g(t, 1)$. By definition, the coefficient of the order of k in this expansion reads

$$g_k(1) = \oint \frac{\mathrm{d}t}{2\pi \mathrm{i} t^{k+1}} g(t, 1) = \int_{1/4}^{+\infty} \frac{\mathrm{d}t}{t^{k+1}} \frac{\mathrm{d}D_X}{\mathrm{d}t} = -\int_0^{\pi/2} (4\cos^2\phi)^{k+1} \frac{\mathrm{d}D_X}{\mathrm{d}\phi}\,\mathrm{d}\phi. \tag{6.8}$$

For large k, the last integral is dominated by small values of ϕ, where $\cos\phi$ can be expanded to quadratic order and exponentiated, while the asymptotic result (6.5) holds for $D_X(\phi)$. It turns out that only the constant Fourier component $B_X^{(0)}$ of the periodic amplitude $B_X(x)$ matters (see below equation (6.10)). We thus obtain

$$g_k(1) \approx 4^{k+1}\, |\ln p|\, B_X^{(0)} \int_0^\infty \phi\, \mathrm{e}^{-k\phi^2 - \pi|\ln p|/\phi}\,\mathrm{d}\phi. \tag{6.9}$$

This integral is then evaluated by the saddle-point approximation, with the saddle-point value of ϕ being $\phi_c = (\pi\,|\ln p|\,/(2k))^{1/3}$. We are thus left with the explicit asymptotic result

$$g_k(1) \approx 4\,|\ln p|^{4/3}\,B_X^{(0)}\left(\frac{\pi}{2k}\right)^{5/6} \exp\left(-\tfrac{3}{2}\left(2\pi^2(\ln p)^2\,k\right)^{1/3}\right)4^k \tag{6.10}$$

which only depends on X through $B_X^{(0)}$ (see equation (6.7)).

It is now clear why only the constant component $B_X^{(0)}$ matters, just as in trapping problems [5,6,10]. Consider the Fourier expansion $B_X(x) = \sum_k B_X^{(k)} e^{2\pi i k x}$. The contribution of the Fourier component $B_X^{(k)}$ is obtained by replacing in equation (6.10) $|\ln p|^{2/3}$ by $(-\ln p - 2\pi i k)^{2/3}$, a number with a strictly greater real part, for any $k \neq 0$. The oscillations of the result (6.5) therefore get damped by exponentials of $k^{1/3}$ in the large-k asymptotics of quantities such as $g_k(1)$.

Coming back to the even-visiting walk problem, using the correspondence between the singularities at $t = \frac{1}{4}$ of $f(t, 1)$ and of $g(t, 1)$ with $p = \frac{1}{2}$, we arrive to the following asymptotic expression for the number $\mathcal{N}_k = f_{2k}(1)$ of closed even-visiting walks of $4k$ steps:

$$\mathcal{N}_k \approx B^{(0)}(2\ln 2)^{4/3}\left(\frac{\pi}{k}\right)^{5/6} \exp\left(-\tfrac{3}{2}\left((2\pi\ln 2)^2\,k\right)^{1/3}\right)2^{4k}. \tag{6.11}$$

Note that the case of even-visiting walks has the symmetry $t \leftrightarrow -t$, so that $f(t, 1)$ has two cuts, at $]-\infty, -\frac{1}{4}]$ and $[\frac{1}{4}, +\infty[$, which cancel each other at odd orders in t, and add up constructively at even orders.

In expression (6.11), $B^{(0)}$ denotes the constant component of the periodic amplitude $B(x)$ of the even-visiting walk problem, which cannot be predicted analytically. The exact enumeration procedure described in section 3 has been carried out up to $k = 457$, i.e.

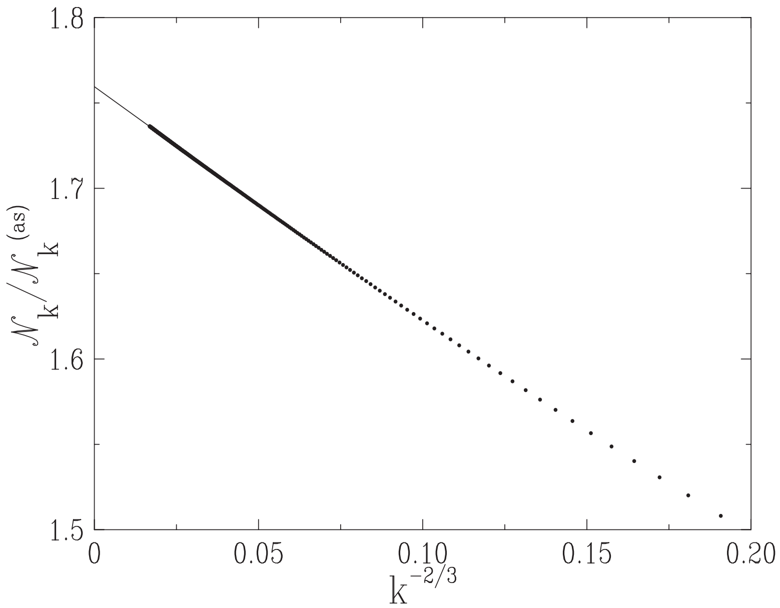

Figure 2. Plot of the numbers $\mathcal{N}_k$ of even-visiting $4k$-steps closed walks, divided by their asymptotic form $\mathcal{N}_k^{(as)}$ defined in the text, against $k^{-2/3}$, up to $k = 457$. Full line, least-squares fit of data with $k \geqslant 50$, with intercept $B^{(0)} = 1.760$.

$4k = 1828$ steps. The number of these walks (exactly evaluated as an integer) reads $\mathcal{N}_{457} \approx 3.368\,295\,75 \times 10^{535}$. Figure 2 shows a plot of the ratio $\mathcal{N}_k$ to its asymptotic behaviour $\mathcal{N}_k^{(\mathrm{as})}$, defined as the result (6.11) without its prefactor $B^{(0)}$. The data exhibit a smooth convergence in $k^{-2/3}$. This form of the leading finite-k correction is indeed expected for all asymptotic estimates, as it corresponds to regular corrections to leading critical behaviour, of relative order $t - t_c$, via the expression of the saddle-point ϕ_c given below equation (6.9). We thus obtain the accurate estimate

$$B^{(0)} \approx 1.760. \tag{6.12}$$

7. Distribution of the maximum height

The purpose of this section is to investigate the distribution of the maximum height reached by an even-visiting walk of $4k$ steps, starting and ending at the origin.

The generating function for even-visiting walks with maximum height M is the difference $f_M(t, 1) - f_{M-1}(t, 1)$, and its coefficient of the order of $2k$ is $f_{M,2k}(1) - f_{M-1,2k}(1)$. Hence the probability that the maximum height in an even-visiting walk of $4k$ steps be M is

$$\Pi_{M,k} = \frac{f_{M,2k}(1) - f_{M-1,2k}(1)}{f_{2k}(1)}. \tag{7.1}$$

The generating function of the cumulants of M therefore reads

$$Z_k(z) \equiv \sum_{j\geqslant 1} \langle\langle M^j \rangle\rangle \frac{z^j}{j!} = \ln\left(\sum_{M\geqslant 0} \Pi_{M,k}\,\mathrm{e}^{zM}\right) = \ln\left(\frac{1-\mathrm{e}^z}{f_{2k}(1)} \sum_{M\geqslant 0} f_{M,2k}(1)\,\mathrm{e}^{zM}\right). \tag{7.2}$$

Let us again make a detour through the random-mirror model. It is straightforward to extend equation (7.2) to the latter situation, by simply replacing fs by gs, and Z by Y, even if the probabilistic interpretation of the formula thus obtained is delicate.

The identity

$$g_N(t, 1) = (1 - p) \sum_{n=0}^{N} p^n Q_n(t)$$

implies

$$Y_k(z) = \ln \frac{c_k(p\,\mathrm{e}^z)}{c_k(p)}$$

with

$$c_k(p) = \oint \frac{\mathrm{d}t}{2\pi \mathrm{i} t^{k+1}} \sum_{n\geqslant 0} p^n Q_n(t) = \frac{g_k(1)}{1-p}.$$

The large-k estimate (6.10) for $g_k(1)$ leads to

$$Y_k(z) \approx \tfrac{3}{2}\,(2\pi^2 k)^{1/3} \left((-\ln p)^{2/3} - (-\ln p - z)^{2/3}\right).$$

Coming back to the even-visiting walk problem, using the correspondence between both problems for $p = \frac{1}{2}$, we obtain

$$Z_k(z) \approx \tfrac{3}{2}\,(4\pi^2 k)^{1/3} \left((\ln 2)^{2/3} - (\ln 2 - z)^{2/3}\right).$$

Hence all the cumulants of the distribution of the maximum height M of $4k$-step walks are asymptotically proportional, namely

$$\langle\langle M^j \rangle\rangle \approx a_j\, k^{1/3} \tag{7.3}$$

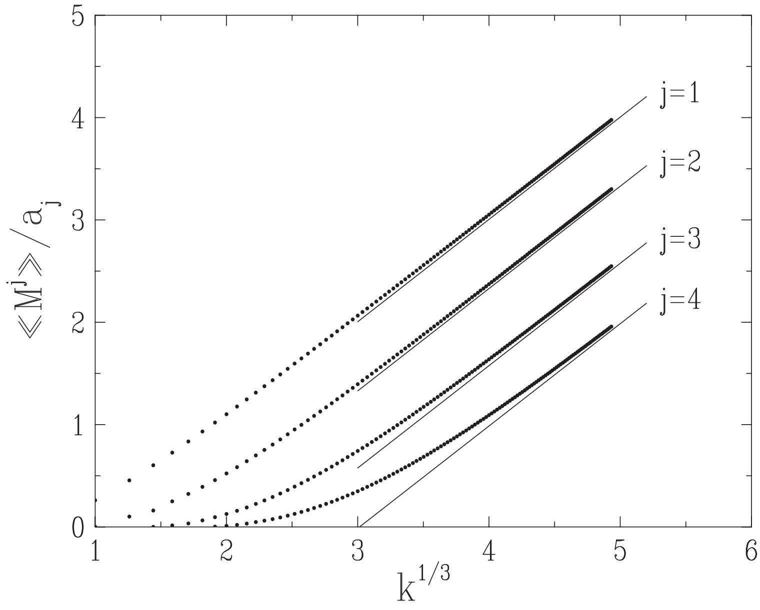

Figure 3. Plot of the first four reduced cumulants $\langle\langle M^j \rangle\rangle/a_j$ of the maximum height M of $4k$-step even-visiting walks, against $k^{1/3}$, up to $k = 125$. The full lines with unit slope demonstrate the quantitative agreement with the analytical results (7.3) and (7.4).

with explicit prefactors

$$a_j = \frac{\Gamma\left(j - \frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} (2\pi)^{2/3} (\ln 2)^{2/3-j} \tag{7.4}$$

i.e. $a_1 = 3.847\,495$, $a_2 = 1.850\,254$, $a_3 = 3.559\,136$, $a_4 = 11.981\,080$ and so on.

As the number of steps becomes large, the distribution of the maximum height M therefore becomes more and more peaked around its mean value $a_1\,k^{1/3}$. The bulk of this distribution is a narrow Gaussian, with standard deviation $a_2^{1/2}\,k^{1/6}$. This result is in sharp contrast with a usual random walk, for which the largest positive extent M scales as $t^{1/2}$, and the ratio $\xi = M/t^{1/2}$ admits a non-trivial limiting probability law [15].

The above analytical results have been checked against data obtained from the exact enumeration procedure of section 3. Figure 3 shows a plot of the first four reduced cumulants $\langle\langle M^j \rangle\rangle/a_j$ $(j = 1, \ldots, 4)$. The clear linear behaviour in $k^{1/3}$, with unit slope, demonstrates a quantitative agreement between the exact data and the asymptotic results (7.3) and (7.4).

8. Amplitude ratios

Up to now, we have only considered averages of Riccati variables with respect to the quenched disorder, such as $f(t, 1)$ and $g(t, 1)$, which have a direct combinatorial interpretation. However, the even-visiting walk model and the random-mirror model also display an interesting dependence on α, which will be investigated in this last section.

To start with, let us consider the random-mirror model with $X = 1$, and assume that $\alpha = \ell$ is a positive integer. We can use equation (6.2) to obtain the pole structure of $Q_n^\ell(t)$, which governs the asymptotics of its t-expansion. As before, it is clear that the pole closest to the critical point ($m = 1$) gives the leading contribution. So, we write

$$Q_n^\ell(t) = \sum_{j=0}^{\ell} \frac{c_j(n,\ell)}{\left(1 - 4t\cos^2[\pi/(n+2)]\right)^j} + \cdots \tag{8.1}$$

where the dots denote a part which is regular at $t = 1/(4\cos^2\frac{\pi}{n+2})$. It turns out that the coefficients $c_j(n,l)$ have a simple behaviour when n is large. Indeed, let us write $\phi = \pi/n - \pi\lambda/n^2 + \mathcal{O}(1/n^3)$, so that $t = 1/(4\cos^2\frac{\pi}{n}) - \pi^2\lambda/(2n^3) + \mathcal{O}(1/n^4)$, and re-express $Q_{n-2}(t)$ in terms of λ. Using equation (5.5), we can check that $Q_{n-2} = 2(1 + 1/\lambda) + \mathcal{O}(1/n)$, so that

$$Q_{n-2}^\ell = 2^\ell\left(1 + \frac{1}{\lambda}\right)^\ell + \mathcal{O}(1/n).$$

On the other hand, a direct substitution of the above expression for t into equation (8.1) yields, for large n,

$$Q_{n-2}^\ell \approx \sum_{j=0}^{\ell} c_j(n-2,\ell)\left(\frac{2\pi^2\lambda}{n^3}\right)^{-j}.$$

Comparing the above two formulae, we obtain, again for large n,

$$c_j(n,\ell) \approx 2^\ell \binom{\ell}{j}\left(\frac{2\pi^2}{n^3}\right)^j.$$

Now, the t-expansion of equation (8.1) is straightforward. A pole of the order of j gives an extra factor $(j+k-1)!/((j-1)!k!)$ compared with a pole of the order of 1. The sum over n is then performed by the saddle-point approximation, with the saddle-point value of n being $n_c \approx (2\pi^2 k/|\ln p|)^{1/3}$. We are thus left with the following explicit values of the amplitude ratios:

$$\mathcal{G}(\ell) \equiv \lim_{k\to\infty} \frac{g_k(\ell)}{g_k(1)} = 2^{\ell-1}\sum_{j=1}^{\ell} \frac{|\ln p|^{j-1}}{(j-1)!}\binom{\ell}{j}. \tag{8.2}$$

It turns out that this result is independent of X, but we shall not go into any more detail.

Quite unexpectedly, the amplitude ratios $\mathcal{G}(\ell)$ are also determined as the solution of an eigenvalue problem. The starting point is equation (5.2), which we divide by $g_m(1)$. Because the radius of convergence of the t-expansion of $g(t,\alpha)$ is $\frac{1}{4}$, we expect that, for large m and fixed n, $g_{m-n}(n) \approx 4^{-n} g_m(n)$. So, we obtain

$$\mathcal{G}(\alpha) = p\sum_{n\geqslant 1} \mathcal{G}(n) 4^{-n} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha) n!}.$$

In particular, fixing $\alpha = \ell$, a positive integer,

$$\mathcal{G}(\ell) = p\sum_{n\geqslant 1} \mathcal{G}(n) 4^{-n} \frac{(\ell+n-1)!}{(\ell-1)!n!}. \tag{8.3}$$

This equation implies that the (infinite) matrix $\mathcal{M}$ with positive entries $\mathcal{M}_{\ell,n} = (\ell+n-1)!/(4^n(\ell-1)!n!)$ for $\ell, n \geqslant 1$ has an eigenvector with eigenvalue $1/p$ and positive components $\mathcal{G}(\ell)$, for every $p \in\,]0,1]$. We therefore have an explicit example of why the Perron–Frobenius theorem (stating that a finite positive matrix has exactly one eigenvector with positive components) cannot extend to infinite matrices without further hypotheses. Note that the matrix $\mathcal{M}$ is highly asymmetric, has unbounded entries, and a divergent trace.

It is interesting to compare the linear system (8.3) with equation (5.1) at $t = \frac{1}{4}$, which reads

$$g\left(\tfrac{1}{4},\alpha\right) = 1 + p\sum_{n\geqslant 1} g\left(\tfrac{1}{4},n\right)4^{-n}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!}.$$

In particular, fixing $\alpha = \ell$, a non-negative integer,

$$g\left(\tfrac{1}{4},\ell\right) = g\left(\tfrac{1}{4},0\right) + p\sum_{n\geqslant 1} g\left(\tfrac{1}{4},n\right)4^{-n}\frac{(\ell+n-1)!}{(\ell-1)!n!}.$$

So, despite the fact that analyticity for $|t| < \frac{1}{4}$ and continuity at $t = \frac{1}{4}$ fix $g\left(\frac{1}{4},\alpha\right)$ unambiguously, the $\mathcal{G}(\ell)$ are a (positive) ambiguity of the solution of equation (5.1) restricted to $t = \frac{1}{4}$. This is again incompatible with the Perron–Frobenius property.

Because of these strange properties, it is reassuring to show that the explicit formula (8.2) for the $\mathcal{G}(\ell)$ indeed satisfies (8.3), which was derived only heuristically. The proof relies on the following identity:

$$\frac{1}{(\ell-1)!}\sum_{j\geqslant 0}\frac{(j+\ell)!}{j!(j+1)!}s^j = \mathrm{e}^s\ell!\sum_{j=1}^{\ell}\frac{s^{j-1}}{(j-1)!j!(\ell-j)!}$$

which holds for ℓ a positive integer, and can be checked by an explicit expansion of its right-hand side in powers of s. We thus obtain the explicit formula

$$\mathcal{G}(\alpha) = p\,\frac{2^{\alpha-1}}{\Gamma(\alpha)}\sum_{j\geqslant 0}\frac{\Gamma(j+\alpha+1)}{j!(j+1)!}\,|\ln p|^j\,. \tag{8.4}$$

When α is large and positive, at fixed $p \neq 1$, the sum over j can be evaluated by the saddle-point approximation, the saddle-point value of j being $j_c \approx (|\ln p|\,\alpha)^{1/2} + |\ln p|/2$, yielding

$$\mathcal{G}(\alpha) \approx \frac{1}{4}\left(\frac{p^2\alpha}{\pi^2\,|\ln p|^3}\right)^{1/4}\exp\left(2(|\ln p|\,\alpha)^{1/2}\right)\,2^{\alpha}. \tag{8.5}$$

In the present case of the random-mirror model, we have been lucky enough to obtain the exact amplitude ratios (8.2). Had this not been the case, equation (8.3) would provide a convenient tool to obtain accurate numerical values for these ratios. In fact, this alternative approach leads to a better understanding of how the Perron–Frobenius theorem is bypassed. If the matrix $\mathcal{M}_{\ell,n}$ is truncated to $\ell, n \leqslant \ell_{\max}$, then, whatever $\ell_{\max}$ is, the spectrum does not contains any eigenvalue larger than unity. So, the right trick is to truncate $\mathcal{M}_{\ell,n}$ to $\ell \leqslant \ell_{\max}$, $n \leqslant \ell_{\max}+1$, to fix the normalization $\mathcal{G}(1) = 1$, and to solve for $\mathcal{G}(2), \ldots, \mathcal{G}(\ell_{\max}+1)$.

Coming back to the even-visiting walk problem, we shall follow the latter strategy. We expect, by analogy with the previous situation, that $\mathcal{F}(\alpha) = \lim_{k\to\infty}(f_{2k}(\alpha)/f_{2k}(1))$ are well behaved limits. Then, starting from equation (3.3), we can repeat the above argument, to show that

$$\mathcal{F}(\alpha) = \sum_{n\geqslant 1}\mathcal{F}(2n)16^{-n}\frac{\Gamma(\alpha+2n)}{\Gamma(\alpha)(2n)!}$$

and, fixing again $\alpha = \ell$, a non-negative integer,

$$\mathcal{F}(\ell) = \sum_{n\geqslant 1}\mathcal{F}(2n)16^{-n}\frac{(\ell+2n-1)!}{(\ell-1)!(2n)!}. \tag{8.6}$$

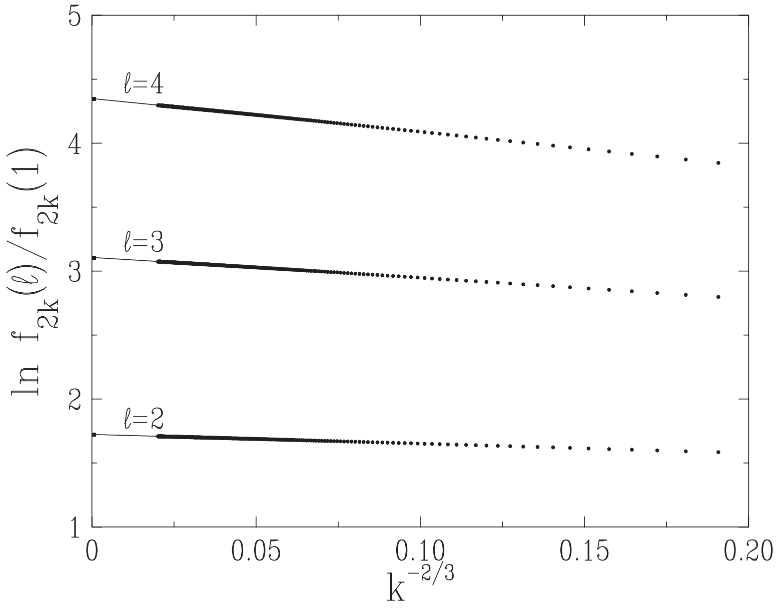

Figure 4. Logarithmic plot of the ratios $f_{2k}(\ell)/f_{2k}(1)$ against $k^{-2/3}$, for $\ell = 2$–4, and k up to 400. Full lines, least-squares fits of data with $k \geqslant 50$. The intercepts match very accurately the predicted limiting amplitude ratios $\mathcal{F}(\ell)$ (symbols on vertical axis).

This result implies that equation (3.1) that determines $f(t, \alpha)$ is ambiguous at $t^2 = \frac{1}{16}$. By truncating the system (8.6) as explained above, we can obtain the amplitude ratios $\mathcal{F}(\ell)$ with very high accuracy, namely $\mathcal{F}(2) = 5.591\,806$, $\mathcal{F}(3) = 22.285\,850$, $\mathcal{F}(4) = 77.059\,126$ and so on.

These predictions have been checked against exact data for the ratios $f_{2k}(\ell)/f_{2k}(1)$, obtained by means of the enumeration approach of section 3. Figure 4 demonstrates a quantitative agreement, for $\ell = 2$–4. The data smoothly converge to the predicted limiting values $\mathcal{F}(\ell)$, with (small) $k^{-2/3}$ corrections. This reinforces our assertion that even-visiting walks and random-mirror models have very similar asymptotic behaviours, up to multiplicative prefactors.

9. Discussion

In this paper we have reconsidered the problem of even-visiting random walks, and obtained several kinds of exact or asymptotic results in one dimension.

We have mapped the even-visiting walk problem onto a non-Hermitian Anderson model (2.5). The weak-disorder expansion of the latter model provides very efficient numerical tools to enumerate and characterize even-visiting walks. We have thus been able to evaluate exactly, among other quantities of interest, the total number $\mathcal{N}_k$ of closed even-visiting walks up to $k = 457$, i.e. $4k = 1828$ steps, thus going far beyond previous works. Indeed, our result is to be compared with the exact result [3] up to 80 steps, and with the approximate numerical simulation [1] up to around 1000 steps.

The mapping to the non-Hermitian Anderson model (2.5) also makes many concepts and techniques of one-dimensional disordered systems available [9, 10]. The analogy with Lifshitz tails, which was already clearly apparent from Derrida's argument [4] in any dimension, has been corroborated at the level of analytical tools in the one-dimensional situation. The escape probability, investigated in section 4, is fully analogous to the integrated density of states in disordered spectra [8–10, 12], with an exponential behaviour, modulated by an oscillatory amplitude, which is asymptotically periodic in the relevant variable. Periodic amplitudes are actually ubiquitous in one-dimensional disordered systems, when the quenched disorder has a discrete distribution [9, 10, 13].

In order to investigate more complex quantities, such as the generating functions of even-visiting walks, we make an extensive use of the random-mirror model. The latter model is similar to the random harmonic chain where a finite fraction of the atoms has an infinite mass, first considered by Domb *et al* [14]: the quantities of interest are directly computable by elementary means. The results thus obtained still hold true for the even-visiting walk problem, the only difference being that the simple and explicit periodic amplitudes of the first model are replaced by unknown periodic functions. The same phenomenon is well established in the case of spectra of one-dimensional disordered systems: details of the distribution of random masses, random site potentials, and so on, only enter the periodic amplitudes of Lifshitz tails [6, 10, 12].

In the present case, just as in trapping problems [5, 6, 10], the quantities of most interest just involve the constant Fourier component of the model-specific periodic amplitudes. For the total number of even-visiting $4k$-step walks, the amplitude (6.12) has been determined very accurately by comparing the data of the exact enumeration procedure to the analytical asymptotic estimate (6.11).

Our analytical investigations have also led to the prediction (7.3) that all the cumulants of the maximum height M reached by an n-step even-visiting walk scale as $n^{1/3}$, with known prefactors. Usual random walks (for which $M/n^{1/2}$ has a non-trivial limit law) and even-visiting ones (for which $M/n^{1/3}$ becomes more and more peaked as n becomes large) are therefore very different in that respect. We have also investigated amplitude ratios, associated with higher moments of the Riccati variables, which exhibit a highly non-trivial dependence in the variable α.

Finally, the asymptotic result (6.11) demonstrates that the outcome (1.2) of Derrida's original argument is very accurate in the one-dimensional case, as it just misses a power of the number of steps (in one dimension we have $\Omega = 2$ and $j = \pi/2$). The analogy with Lifshitz tails also suggests that the situation is less under control in higher dimension: going beyond the leading estimate (1.2), which involves only the volume of the Lifshitz sphere, indeed remains a difficult open problem [16].

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